CS 70 Discrete Mathematics and Probability Theory
Fall 2025 Sabin, Hug

DIS 2B

Graph Theory I

Note 5

A graph $G = (V,E)$ consists of a set of vertices V and a set of pairs of vertices $(u,v) \in E$ with $u,v \in V$. In a directed graph, an edge $(u,v) \in E$ is directed from u to v. In an undirected graph the pair is unordered. Unless otherwise specified, graphs in this class are undirected and simple (no self-loops or multiple edges).

Degree: An edge (u,v) is incident to u and v. The degree of a vertex v is the number of edges incident to it, denoted $\deg(v)$.

Degree-sum Formula: $\sum_{v \in V} \deg(v) = 2|E|$. The total number of edge vertex incidences is the sum of the degrees by definition of degree, and also twice the number of edges as each edge is incident to 2 vertices.

Path: A sequence of edges with no repeated vertices. Formally, there is a path between u and v when there is a sequence of vertices $u = v_0, \ldots, v_k = v$ where successive vertices are in an edge, i.e., $(v_i, v_{i+1}) \in E$.

Walk: A sequence of edges $\{(v_1,v_2),(v_2,v_3),\ldots(v_{n-1},v_n)\}$ with possibly repeated vertices.

Cycle: A sequence of edges with start = end, and no other repeated vertices.

Tour: A sequence of edges with start = end, and there may be repeated vertices.

Eulerian Tour: A tour that uses every edge in graph exactly once.

Connected: (u,v) are connected in $G = (V,E)$ if there is a path between u and v. A graph is connected if all pairs of vertices are connected.

Bipartite graph: A graph G with two groups of vertices such that all edges are incident to one vertex in each group.

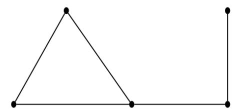
Tree: A graph is a tree iff it satisfies any of the following:

- connected and acyclic
- connected and has $|V| - 1$ edges
- connected, and removing any edge disconnects the graph
- acyclic, and adding any edge creates a cycle

1 Degree Sequences

Note 5

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3,2,2,2,1)$.



For each of the parts below, determine if there exists a simple undirected graph G (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

(a) $(3,3,2,2)$

(b) $(3,2,2,2,2,1,1)$

(c) $(6,2,2,2)$

(d) $(4,4,3,2,1)$

2 Build-Up Error?

Note 5

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

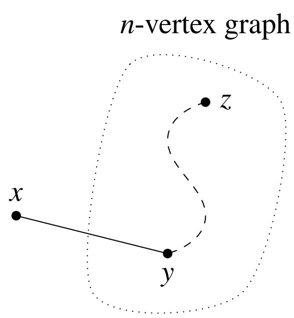
False Claim: If every vertex in an undirected graph with $|V| \geq 2$ has degree at least 1, then it is connected.

Proof? We use induction on the number of vertices $n \geq 2$.

Base case: The only valid graph has two vertices joined by an edge. This graph is connected, so the base case is true.

Inductive hypothesis: Assume the claim is true for some $n \geq 2$.

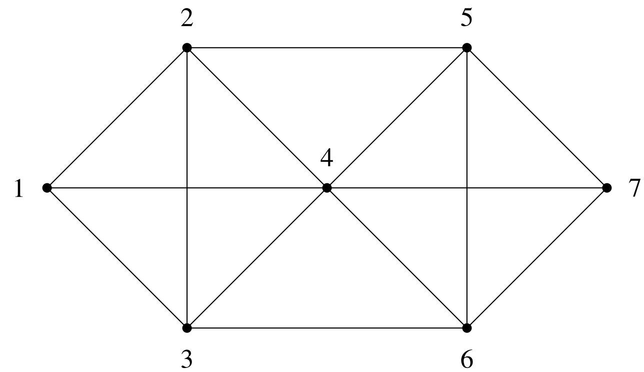
Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$. $\square$

3 Eulerian Tour and Eulerian Walk

Note 5



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

4 Coloring Trees

Note 5

(a) Prove that all trees with at least 2 vertices have at least two leaves. Recall that a leaf is defined as a node in a tree with degree exactly 1.

(b) Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]