Discrete Probability Intro

Note 13

Probability Space: A probability space is a tuple $(\Omega, \mathbb{P})$, where Ω is the *sample space* and $\mathbb{P}$ is the *probability function* on the sample space.

Specifically, Ω is the set of all outcomes ω, and $\mathbb{P}$ is a function $\mathbb{P}\colon \Omega \to [0,1]$, assigning a probability to each outcome, satisfying the following conditions:

$$0 \le \mathbb{P}[\omega] \le 1 \quad \text{and} \quad \sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1.$$

Event: an event A is a subset of Ω, i.e. a collection of some outcomes in the sample space. We define

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega].$$

Uniform Probability Space: all outcomes are assigned the same probability, i.e. $\mathbb{P}[\omega] = \frac{1}{|\Omega|}$; this is just counting!

With an event A in a uniform probability space, $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$, which is again more counting!

1 Flippin' Coins

Note 13

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

(g) What is the probability of the event that our outcome has at least one head?

2 Sampling

Note 13

Suppose you have balls numbered $1,\ldots,n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

3 Monty Hall Variant

Note 14

Recall the Monty Hall problem introduced in lecture. You are on a gameshow with 3 doors, behind which there are 2 goats and 1 new car. You choose a door. Then, the host reveals one door, which is a goat. Now, they ask if you if you would like to switch the door you have chosen to the other remaining one. Do you switch? Turns out, switching is the best choice from a probabilistic standpoint! Let's explore why this is and look at some modifications.

(a) After many years, the standard version of the game with three doors becomes a little boring, so Monty decides to increase the number of doors to four (with one prize and three goats). What is now the probability that the contestant wins under the switching strategy?

(b) Monty Hall gets bored with the normal game with three doors, so he decides to increase the number of doors to five (one prize door and four with goats behind them). This time after the contestant selects a door, at least three of the remaining doors must have goats behind them. Monte picks at random two of the remaining doors with goats behind them and opens them. He then gives the contestant the option of switching to one of the two unopened doors.

What is the chance the contestant wins under the switching strategy? What is the chance the contestant wins under the sticking strategy? Should the contestant switch?

4 Intransitive Dice

Note 13

You're playing a game with your friend Bob, who has a set of three dice. You'll each choose a different die, roll it, and whoever had the higher result wins. The dice have sides as follows:

- Die A has sides 2, 2, 4, 4, 9, and 9.
- Die B has sides 1, 1, 6, 6, 8, and 8.
- Die C has sides 3, 3, 5, 5, 7, and 7.

(a) Suppose you have chosen die A and Bob has chosen die B. What is the probability that you win?
Hint: It may be easier to work with a sample space smaller than 6×6.

(b) Suppose you have chosen die B and Bob has chosen die C. What is the probability that you win?

(c) Suppose you have chosen die C and Bob has chosen die A. What is the probability that you win?

(d) Bob offers to let you choose your die first so that you can choose the best one. Is this an offer your should accept? Why or why not?