Due: Saturday, 9/13, 4:00 PM
Grace period until Saturday, 9/13, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

1 Induction Starter

Note 3

Consider the inequality $2^n < n!$ (the right hand side is a factorial, not an exclamation mark).

(a) Make a conjecture as to which $n \in \mathbb{N}$ will have the inequality hold.

We will now prove your conjecture using induction.

(b) What is your base case?

(c) What is the inductive hypothesis for this proof?

(d) What do we want to show in the inductive step?

(e) Conclude the proof with the inductive step.

2 Airport

Note 3

Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

3 Proving Inequality

Note 3

For all positive integers $n \geq 1$, prove with induction that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we require you to use induction. If direct induction seems difficult, consider strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

4 Universal Preference

Note 4

Suppose that preferences in a stable matching instance are universal: all n jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What pairing do we get from running the algorithm with jobs proposing (Hint: Start with small examples and go through the algorithm. Do you see a pattern?)? Prove that this happens for all n.

(b) What pairing do we get from running the algorithm with candidates proposing? Explain.

(c) What does this tell us about the number of stable pairings? Justify your answer.

5 Pairing Up

Note 4

Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

(*Hint*: It can help to start with some small examples; find an instance for $n = 2$, and think about how you can use these preference lists to construct an instance for $n = 4$. After this, you should be in a good position to generalize the construction for all even n. Additionally, $2^{n/2}$ is a very specific number; try to think about how your construction would build such a number as it is constructed with increasing n.)

6 Optimal Candidates

Note 4

In the notes, we proved that the propose-and-reject algorithm always outputs the job-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every job with its optimal candidate results in a pairing at all. Prove by contradiction that no two jobs can have the same optimal candidate. (Note: your proof should not rely on the fact that the propose-and-reject algorithm outputs the job-optimal pairing.)