CS 70	Discrete Mathematics and Probability Theory	
Fall 2025	Sabin, Hug	HW 03

Due: Saturday, 9/20, 4:00 PM
Grace period until Saturday, 9/20, 6:00 PM
Remember to show your work for all problems!

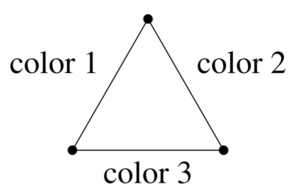
Sundry

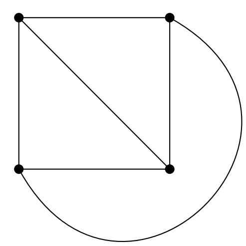
Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

1 Edge Colorings

Note 5

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (You may use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Let $d \geq 1$. Prove that any graph with maximum degree d can be edge colored with $2d-1$ colors.

(c) Prove that a tree can be edge colored with d colors where d is the maximum degree of any vertex. *Hint*: Different problems sometimes are easier to induct over different parameters even when the problems are very similar.

2 Planarity and Graph Complements

Note 5

Let $G=(V,E)$ be an undirected graph. We define the complement of G as $\overline{G}=(V,\overline{E})$ where $\overline{E}=\{(i,j) \mid i,j \in V, i \neq j\}-E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

3 Proofs in Graphs

Note 5

(a) Suppose California has n cities ($n \geq 2$) such that for every pair of cities X and Y, either X has a road to Y or Y has a road to X. Further, suppose that all roads are one-way streets.

Prove that regardless of the configuration of roads, there always exists a city which is reachable from every other city by traveling through at most 2 roads.

[*Hint:* Induction]

(b) Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

[*Hint:* In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof.]

(c) Prove that any graph G is bipartite if and only if it has no tours of odd length.

[*Hint:* In one of the directions, consider the lengths of paths starting from a given vertex.]

4 Touring Hypercube

Note 5

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph (with ≥ 2 vertices) is a tour that visits every vertex exactly once (except that the start and end vertices are the same).

(a) Prove that a hypercube has an Eulerian tour if and only if n is even.

(b) Prove that every hypercube has a Hamiltonian tour.

5 (Optional) Anonymous Feedback Form

Please fill out this optional form to let us know how you are doing in CS 70 and how you are absorbing the course material: `https://forms.gle/whoHyv64omndEWhd7`!