

Due: Saturday, 11/8, 4:00 PM  
Grace period until Saturday, 11/8, 6:00 PM  
Remember to show your work for all problems!

## Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

## 1 Combined Head Count

Note 18

Suppose you flip a fair coin twice.

- (a) What is the sample space  $\Omega$  generated from these flips?
- (b) Define a random variable  $X$  to be the number of heads. What is the distribution of  $X$ ?
- (c) Define a random variable  $Y$  to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of  $Y$ ?
- (d) Compute the conditional probabilities  $\mathbb{P}[Y = i \mid X = j]$  for all combinations of  $i$  and  $j$ .
- (e) Define a third random variable  $Z = X + Y$ . Use the conditional probabilities you computed in part (d) to find the distribution of  $Z$ .

## 2 Testing Model Planes

Note 18

Amin is testing model airplanes. He starts with  $n$  model planes which each independently have probability  $p$  of flying successfully each time they are flown, where  $0 < p < 1$ . Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying all remaining model planes and throwing away any that crash. Let  $X_i$  be the random variable representing how many model planes remain after  $i$  days. Note that  $X_0 = n$ . Justify your answers for each part.

- (a) What is the distribution of  $X_1$ ? That is, what is  $\mathbb{P}[X_1 = k]$ ?
- (b) What is the distribution of  $X_2$ ? That is, what is  $\mathbb{P}[X_2 = k]$ ? Recognize the distribution of  $X_2$  as one of the famous ones and provide its name and parameters.

- (c) Repeat the previous part for  $X_t$  for arbitrary  $t \geq 1$ .
- (d) What is the probability that at least one model plane still remains (has not crashed yet) after  $t$  days? Do not have any summations in your answer.
- (e) Considering only the first day of flights, is the event  $A_1$  that the first and second model planes crash independent from the event  $B_1$  that the second and third model planes crash? Recall that two events  $A$  and  $B$  are independent if  $\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$ . Prove your answer using this definition.
- (f) Considering only the first day of flights, let  $A_2$  be the event that the first model plane crashes *and* exactly two model planes crash in total. Let  $B_2$  be the event that the second plane crashes on the first day. What must  $n$  be equal to in terms of  $p$  such that  $A_2$  is independent from  $B_2$ ? Prove your answer using the definition of independence stated in the previous part.
- (g) Are the random variables  $X_i$  and  $X_j$ , where  $i < j$ , independent? Recall that two random variables  $X$  and  $Y$  are independent if  $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1] \mathbb{P}[Y = k_2]$  for all  $k_1$  and  $k_2$ . Prove your answer using this definition.

### 3 Fishy Computations

Note 18

Assume for each part that the random variable can be modelled by a Poisson distribution.

- (a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?
- (b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2024?
- (c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?
- (d) Denote  $X \sim \text{Pois}(\lambda)$ . Prove that

$$\mathbb{E}[Xf(X)] = \lambda \mathbb{E}[f(X+1)]$$

for any function  $f$ .

### 4 Such High Expectations

Note 19

Suppose  $X$  and  $Y$  are independently drawn from a Geometric distribution with parameter  $p$ . For each of the below subparts, your answer must be simplified (i.e. NOT left in terms of a summation).

- (a) Compute  $\mathbb{E}[\min(X, Y)]$ .
- (b) Compute  $\mathbb{E}[\max(X, Y)]$ .
- (c) Compute  $\mathbb{P}[X + Y \geq t]$

## 5 Swaps and Cycles

Note 19

A permutation of  $n$  objects is a bijection from  $(1, \dots, n)$  to itself. For example, the permutation  $\pi = (2, 1, 4, 3)$  of 4 objects is the mapping  $\pi(1) = 2$ ,  $\pi(2) = 1$ ,  $\pi(3) = 4$ , and  $\pi(4) = 3$ . We'll say that a permutation  $\pi = (\pi(1), \dots, \pi(n))$  contains a *swap* if there exist  $i, j \in \{1, \dots, n\}$  so that  $\pi(i) = j$  and  $\pi(j) = i$ , where  $i \neq j$ . The example above contains two swaps:  $(1, 2)$  and  $(3, 4)$ .

- (a) In terms of  $n$ , what is the expected number of swaps in a random permutation?
- (b) In the same spirit as above, we'll say that  $\pi$  contains a *k-cycle* if there exist  $i_1, \dots, i_k \in \{1, \dots, n\}$  with  $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_k) = i_1$ . Compute the expectation of the number of  $k$ -cycles.